



Master's thesis
Theoretical Physics

Electron and Heat Transport through Low-Dimensional Quantum Structures

Pasi Ritaluoto

April 10, 2018

Tutor: doc. Esko Keski-Vakkuri

Censors: doc. Esko Keski-Vakkuri
prof. Kari Rummukainen

UNIVERSITY OF HELSINKI
DEPARTMENT OF PHYSICS

PL 64 (Gustav Hållströmin katu 2)
00014 Helsingin yliopisto

Tiedekunta — Fakultet — Faculty		Laitos — Institution — Department	
Faculty of Science		Department of Physics	
Tekijä — Författare — Author			
Pasi Ritaluoto			
Työn nimi — Arbetets titel — Title			
Electron and Heat Transport through Low-Dimensional Quantum Structures			
Oppiaine — Läroämne — Subject			
Theoretical Physics			
Työn laji — Arbetets art — Level		Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Master's thesis		April 10, 2018	44 pages
Tiivistelmä — Referat — Abstract			
<p>In this thesis the transport properties of the superconducting quantum wires are discussed. At first the general introduction to quasi-one-dimensional mesoscopic systems is given. Due to the spatial confinement of transported particles in directions vertical to their motion, there are possibly many discrete transverse modes along the wire. The number of these modes depends on the width of the wire.</p> <p>According to the familiar Ohm's law, the conductance along the wire has linear dependence on the width of the wire. However, when the size of the system is sufficiently small, there exist quantum corrections to this behaviour. The leading order quantum corrections to the Ohm's law are the weak localization and universal conductance fluctuations.</p> <p>These quantum corrections can be determined by calculating the first two conductance cumulants along the wire. In addition, there emerges terms in these cumulants, which depend nonanalytically on the wire length. In this thesis we calculate all these terms by using the diffusion type equation, Dorokhov-Mello-Pereyra-Kumar (DMPK) equation. We calculate also the analytical components of the first 15 current cumulants through the system.</p>			
Avainsanat — Nyckelord — Keywords			
Mesoscopic systems, transport, random matrices, cumulants			
Säilytyspaikka — Förvaringsställe — Where deposited			
Muita tietoja — övriga uppgifter — Additional information			

Contents

1	Introduction	1
2	Physical backgrounds	3
2.1	Scattering and transfer matrices	3
2.2	Landauer formula	4
2.3	Superconducting quantum wires	6
2.4	Interference phenomena	7
2.4.1	Weak localization	7
2.4.2	Universal conductance fluctuations	9
2.5	Spin-rotation invariance	10
2.5.1	The Foldy-Wouthuysen Transformation	11
2.6	Time-reversal Symmetry	12
3	Theory	14
3.1	Random matrix theory	14
3.2	Classification of symmetry classes of the random matrix theories . . .	16
3.2.1	Symmetry class D	16
3.2.2	Symmetry class C	17
3.2.3	Symmetry class DIII	17
3.2.4	Symmetry class CI	18
3.3	Dorokhov-Mello-Pereyra-Kumar equation	18

3.4	Conductance cumulants	20
3.5	Current cumulants	21
3.6	Calculation of expectation values	23
3.6.1	Second moment	23
3.6.2	Third moment	24
3.7	Alternative form of the DMPK equation	24
4	Calculations	25
4.1	Calculation of the correlation function	25
4.2	Three first moments of the conductance	28
4.2.1	First Moment	28
4.2.2	Second Moment	32
4.2.3	Third moment	34
4.3	Moments in Different Regimes	34
4.3.1	Metallic Regime	34
4.3.2	Localized Regime	36
4.4	Current cumulants in metallic regime	36
5	Results	39
6	Discussion	41
	Bibliography	43
	References	43

1. Introduction

During the last decades, electron and heat transport in *mesoscopic* systems have gained a lot of interest. The term mesoscopic refers to systems whose typical dimensions have an intermediate length between the microscopic and macroscopic scales ranging from a few nanometers to hundreds of micrometers. One interesting feature of those systems is that they serve as a playground to experimentalists and theorists to investigate physical phenomena observed earlier in metals. Usually mesoscopic systems are in some sense more flexible than the corresponding metallic systems, which means that some parameters, such as Fermi energy, can be adjusted in those systems.

The size of the mesoscopic systems contributes drastically on their physical properties. There are three especially important characteristic lengths in mesoscopic systems. 1) The de Broglie wavelength is the length, which is related to the quantum mechanical wave nature of the electron. It determines whether the system is so small that the quantum effects become important. 2) The electron mean free path is the average distance that an electron travels between two successive collisions of the electron with the impurities. 3) The electron phase relaxation length is the average length that electron travels before its phase is destroyed.

In this Thesis we investigate systems, in which the size of the system is less than de Broglie wavelength and phase relaxation length of the electrons. We also focus on quasi-one-dimensional systems, whose length is much greater than their

width and the system can thus be thought as a wire with a definite length L . If the electron mean free path l obeys $L \ll l$, we say that the system is in the ballistic regime. If $l \ll L \ll Nl$, where N is the number of transverse modes in the wire, we say that the system is in the diffusive or metallic regime. Finally, if $L \gg Nl$, we say that the system is in the localized regime. In each regime, transport properties of the system as a function of L can be determined by using the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation. This approach is described more precisely in the following chapters.

In this Thesis we have proposed a new approach to calculate certain integrals involving Jacobi polynomials, and used that to derive an exact formula, from which the analytical terms of the current cumulants through the superconducting quantum wires can be evaluated up to the arbitrary order. We have also computed analytical terms of the first 15 current cumulants explicitly.

2. Physical backgrounds

In this chapter we introduce some concepts which are used extensively in later chapters.

2.1 Scattering and transfer matrices

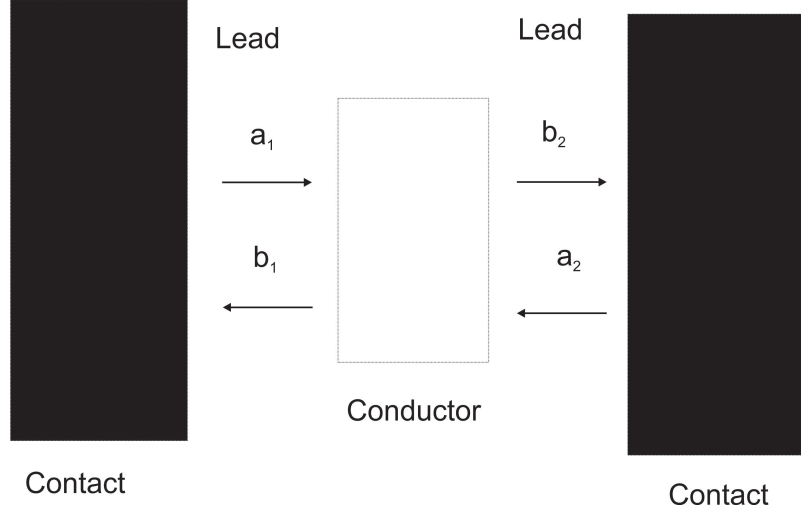
We have a mesoscopic sample (e.g. a quantum wire) sandwiched between two leads, which are coupled to two contacts. The density of states (DOS) of the electrons is continuous in the contacts, whereas there are discrete transverse modes in the leads. The device geometry is shown in Figure 2.1.

The amplitudes of the wavefunctions of the incoming and outgoing electrons in the left (right) lead are contained in the components of the column vectors a_1 (a_2) and b_1 (b_2). The different modes are related through the scattering matrix as follows:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad (2.1)$$

where the matrices $r(r')$ and $t(t')$ describe reflection and transmission processes of the electrons arising from the left (right) lead. The transfer matrix M on the other hand, relates the amplitudes in the left lead to those in the right lead as follows:

$$\begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = M \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \quad (2.2)$$

**Figure 2.1:** Device geometry

where [1]

$$M = \begin{bmatrix} (t^\dagger)^{-1} & r'(t')^{-1} \\ -(t')^{-1}r & (t')^{-1} \end{bmatrix}, \quad (2.3)$$

where the unitarity of the scattering matrix due to the flux conservation was used. The transfer matrix has a product composition property which means that the transfer matrix of the system consisting of two or more adjacent samples is the matrix product of transfer matrices of individual scatterers.

2.2 Landauer formula

Conductance through a quantum wire can be calculated by using the Landauer formalism which has now become standard in the description of nanostructure transport (For a review, see, for example, Ref. [2]).

Let's consider the transport through a conductor connected to two large contacts through the two leads as shown in Fig. 2.1. When only the lowest subband (channel) is occupied, the current injected from the left and right may be written

as [3]:

$$I = \frac{2e}{2\pi} \left[\int_0^\infty dk v(k) f_L(k) T(E) - \int_0^\infty dk v(k') f_R(k') T(E') \right], \quad (2.4)$$

where the constant is the one-dimensional (1D) DOS in k-space in which the spin degeneracy is taken into account, $v(k)$ is the velocity, $T(E)$ is the transmission coefficient, and f_L and f_R are the distribution functions of the contacts characterized by chemical potentials μ_L and μ_R , respectively. The integrations are only over positive k and k' relative to the direction of the injected charge. If we now assume a low temperature, we can approximate the distribution functions f_L and f_R by step functions and therefore electrons are injected up to an energy μ_L into the left lead and are injected up to μ_R into the right one. Converting integrals over k to integrals over energy, the current becomes

$$\begin{aligned} &= \frac{2e}{2\pi} \left[\int_0^{\mu_L} dE \left(\frac{dk}{dE} \right) v(k) f_L(k) T(E) - \int_0^{\mu_R} dE \left(\frac{dk'}{dE} \right) v(k') f_R(k') T(E) \right] \\ &= \frac{2e}{2\pi\hbar} \int_{\mu_R}^{\mu_L} dE T(E), \end{aligned} \quad (2.5)$$

where we have used the fact that the group velocity of a wave-packet describing the electron in 1D is given by $v = \frac{1}{\hbar} \frac{dE}{dk}$ [4]. If we further assume that the applied voltage is small (i.e., in the linear response regime) so that the energy dependence of $T(E)$ is negligible, the current becomes simply

$$I = \left(\frac{2e}{h} \right) T(\mu_L - \mu_R). \quad (2.6)$$

Since the applied voltage V over the system satisfies $eV = \mu_L - \mu_R$, the conductance through the system takes the form

$$G = \frac{I}{V} = \left(\frac{2e^2}{h} \right) T \quad (2.7)$$

which is finally the single-channel Landauer formula [2]. Landauer's formula thus states that even when there is no backscattering in the system, i.e. when the transmission is unity, there is still contact resistance and thus G obtains a finite value which is the quantum of conductance $\frac{2e^2}{h}$.

The above treatment can be generalized to the multichannel case in which multiple independent channels are present. The conductance is then given by [3]

$$G = \frac{2e^2}{h} \sum_{ij}^N |t_{ij}|^2, \quad (2.8)$$

where $|t_{ij}|^2$ is the probability that an incoming wave at mode i in the left lead is transmitted into mode j in the right lead. N is the number of occupied channels which is given for the wire of width W by [5]

$$N = \text{Int} \left[\frac{k_F W}{\pi} \right], \quad (2.9)$$

where $\text{Int}(x)$ represents the integer that is just smaller than x and k_F is the common Fermi wavenumber of the leads.

Eq. (2.8) can be rewritten as [3]

$$G = G_0 \text{tr} t t^\dagger, \quad (2.10)$$

where $G_0 = \frac{de^2}{h}$ and d is the multiplicity of the eigenvalues of the product $t t^\dagger$ of the transmission submatrices.

2.3 Superconducting quantum wires

Almost the same machinery as in the previous section can be used to study the heat and spin transport in dirty superconducting wires. Unconventional superconductors can be described by a Hamiltonian of Bogoliubov-deGennes (BdG) type. We distinguish gradings corresponding to spin up/down, particle/hole, left/right movers. Denoting these with Pauli matrices σ , γ and τ , where Pauli Spin matrices are given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (2.11)$$

respectively, we write our model Hamiltonian as [6]

$$\mathcal{H} = \mathcal{K} + \mathcal{V}, \quad \mathcal{K} = i v_F \partial_x \sigma_0 \otimes \gamma_0 \otimes \tau_3 \otimes I_N, \quad (2.12)$$

where σ_0 (γ_0) is the 2×2 unit matrix in the spin (particle/hole) grading. The kinetic energy \mathcal{K} describes the propagation of right and left moving quasiparticles in N channels at the Fermi level. The “potential” $\mathcal{V}(x)$ is an $8N \times 8N$ matrix that accounts both the presence of disorder and of superconducting correlations. In particle/hole (γ) grading it reads

$$\mathcal{V} = \begin{pmatrix} v & \Delta \\ -\Delta^* & -v^T \end{pmatrix} \quad (2.13)$$

where v (Δ) is a Hermitian (antisymmetric) $4N \times 4N$ matrix representing the impurity potential (superconducting order parameter). The Hamiltonian is related to the transfer matrix through the following relation:

$$\mathcal{M}(x+L, x) = T_y \exp \left[i \int_x^{x+L} dy I_4 \otimes \tau_3 \otimes I_N \mathcal{V}(y) \right], \quad (2.14)$$

where the T_y denotes the path ordering operator for the y integration along the wire.

It should be stressed that the BdG Hamiltonian does not conserve charge. Instead, the conserved densities are those of the energy (in all four classes) and spin (when spin rotation symmetry is present). Thus, the transport properties (the conductance G) studied in superconducting wires refer to transport of heat and spin.

2.4 Interference phenomena

We mentioned in the introduction, that the size of the system is less than the phase relaxation length of the electrons traversing through the system. That implies that certain interference phenomena may occur in the sample. Below we investigate these phenomena in greater detail.

2.4.1 Weak localization

The weak localization is the general phenomenon occurring in mesoscopic samples, whose impurities are located randomly. Its origin can be traced back to the enhanced

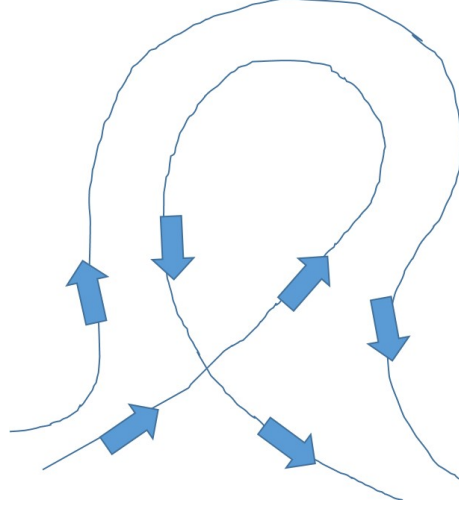


Figure 2.2: Two interfering paths on a loop.

backscattering of electrons along the paths starting and ending to the same mode, as shown in Figure 2.2. Then the probability amplitude of two interfering paths is not the incoherent sum of probabilities of the paths $2|A|^2$ but the quantum coherent probability $|A + A|^2 = 4|A|^2$. Thus the quantum mechanical probability for return is twice the classical value. As we will see, external magnetic field destroys the time-reversal symmetry and the weak localization effect. When spin-orbit scattering is strong and there is no external magnetic field, the spin of the electron changes along the path, the electron paths interfere destructively and conductance is increased. Then instead of weak localization, weak antilocalization occurs.

The weak localization correction to the conductance was first predicted by Gor'kov et. al. by using diagrammatic techniques [7]. By using an approach based on the DMPK equation, Imamura and Hikami [8] calculated the first order correction to the conductance appropriate for all symmetry classes presented in Table (3.1) and thus extended the results by Gor'kov et. al.. Their result was

$$\langle G/G_0 \rangle \approx \frac{1}{s} + \frac{\beta - 2\alpha}{3\beta}, \quad (2.15)$$

where $s = L/Nl$ is the dimensionless length of the wire and $G_0 = \frac{de^2}{h}$ is the conductance quantum. The meaning of the remaining parameters α and β is explained in

Table (3.1).

The first term in Eq. 2.15 is the familiar Ohm's law. The weak (anti-) localization correction to the conductance is given by the second term.

2.4.2 Universal conductance fluctuations

We remind that the conductance through the sample is given by the Eq. (2.8) :

$$G = G_0 \sum_{i,j=1}^N |t_{ij}|^2. \quad (2.16)$$

Let us rewrite Eq. (2.16) as

$$G = G_0 \left[N - \sum_{i,j=1}^N |r_{ij}|^2 \right]. \quad (2.17)$$

The variance of the conductance is given simply by

$$\text{var}(G) = G_0^2 \text{var} \left(\sum_{i,j=1}^N |r_{ij}|^2 \right) = G_0^2 N^2 \text{var}(|r_{ij}|^2), \quad (2.18)$$

with uncorrelated reflection coefficients. Each of the quantities $|r_{ij}|^2$ can be obtained by squaring the amplitudes A_P for the all possible Feynman paths starting in mode i and ending in mode j : [5]

$$|r_{ij}|^2 = \left| \sum_P A_P \right|^2. \quad (2.19)$$

If we assume that the phases of the amplitudes A_P of all paths are completely random then

$$\langle |r_{ij}|^2 \rangle = \sum_P \sum_{P'} \langle A_P A_{P'}^* \rangle = \sum_P |A_P|^2, \quad (2.20)$$

where the brackets denote the ensemble average over all possible impurity configurations, while

$$\begin{aligned} \langle |r_{ij}|^4 \rangle &= \sum_{P,P',P'',P'''} \langle A_P A_{P'} A_{P''}^* A_{P'''}^* \rangle \\ &= \sum_{P,P',P'',P'''} |A_P|^2 |A_{P'}|^2 [\delta_{P,P''} \delta_{P',P'''} + \delta_{P,P'''} \delta_{P',P''}] \\ &= 2 \langle |r_{ij}|^2 \rangle^2. \end{aligned} \quad (2.21)$$

Hence

$$\text{var}(|r_{ij}|^2) = \langle |r_{ij}|^4 \rangle - \langle |r_{ij}|^2 \rangle^2 = \langle |r_{ij}|^2 \rangle^2. \quad (2.22)$$

For a sample that is many mean free paths long,

$$\langle |r_{ij}|^2 \rangle \approx 1/N, \quad (2.23)$$

so that

$$\text{var}(G) \approx G_0^2. \quad (2.24)$$

The more accurate treatment (see e.g. Ref. [8]) shows, that

$$\text{var}(G) \approx G_0^2 \frac{2}{15\beta}. \quad (2.25)$$

This term, which is independent of the length of the wire, is known as universal conductance fluctuations.

As we will see in the later sections, the transport properties of the system depend strongly on the presence or absence of spin-rotation invariance and time-reversal symmetry in the system. Below we discuss these effects in more detail.

2.5 Spin-rotation invariance

When the interactions may cause a spin of an electron to flip, the spin degree of freedom must be explicitly taken into account in the calculations. The spin interacts with the environment through various interactions. A spin may flip due to a spin-dependent interaction with impurities, boundaries, interfaces and phonons or due to a strong spin-orbit scattering [9].

To see, how this spin-orbit interaction emerges, we first study the relativistic version of the Schrödinger equation, the Dirac equation. It describes the motion of the charged particle in the presence of the electromagnetic field caused by the scalar potential V and vector potential \mathbf{A} as follows:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi, \quad (2.26)$$

where ψ is the four component wave vector of the particle and the Hamiltonian operator H is given by [10]:

$$H = c\boldsymbol{\alpha} \cdot (\mathbf{p} - \frac{e}{c}\mathbf{A}) + \beta mc^2 + eV I_4, \quad (2.27)$$

where I_4 denotes 4×4 unit matrix. Here

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (2.28)$$

where the unit entries in β stand for 2×2 unit matrices. Matrices α_i and β obey the following anticommutation relations:

$$\begin{aligned} \{\alpha_i, \alpha_j\} &= 2\delta_{ij} I_4, \\ \{\alpha_i, \beta\} &= 0, \\ \beta^2 &= I_4. \end{aligned} \quad (2.29)$$

We will work in the Schrödinger picture in which particle states are time dependent whereas operators are not.

2.5.1 The Foldy-Wouthuysen Transformation

In wave vector ψ two uppermost components describe the two spin states of the positive energy particle whereas two lowermost components describe the two spin components of the negative energy hole. We call the operators as even if they do not couple particle and hole states, otherwise they are odd. Thus, for instance β and I are even operators whereas α_i 's are odd. The idea of Foldy-Wouthuysen transformation is to transform the four-component Dirac equation to two two-component wave equations, one for particles and another for holes. This can be done by treating the odd operator part in H as small perturbation to the even operator part of H by performing canonical transform that removes odd operator part from the Hamiltonian to certain order.

To be more specific, we rewrite the Hamiltonian H as

$$H = \beta mc^2 + \mathcal{O} + \mathcal{E}, \quad (2.30)$$

where $\mathcal{O} = c\boldsymbol{\alpha} \cdot (\mathbf{p} - \frac{e}{c}\mathbf{A})$ and $\mathcal{E} = eVI_4$. Then we find that

$$\{\beta, \mathcal{O}\} = 0 \quad (2.31)$$

and

$$[\beta, \mathcal{E}] = 0. \quad (2.32)$$

The Hamiltonian H is then transformed as follows:

$$\begin{aligned} H' = e^{iS} H e^{-iS} &= H + i[S, H] + \frac{i^2}{2}[S, [S, H]] + \dots \\ &+ \frac{i^n}{n!}[S, [S, \dots, [S, H] \dots]] + \dots \end{aligned} \quad (2.33)$$

The commutation and anticommutation identities (2.31) and (2.32) can be used to construct operator S that removes odd operators from Hamiltonian H' to certain order in $1/m$, which is small in the nonrelativistic limit.

After some labour, by performing three canonical transformations and by setting $c = \hbar = 1$, we finally find that the transformed Hamiltonian H''' is given by

$$\begin{aligned} H''' &= \beta \left(m + \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} - \frac{\mathbf{p}^4}{8m^2} \right) + eVI_4 - \frac{e}{2m}\beta\boldsymbol{\sigma} \cdot \mathbf{B} \\ &+ \frac{e}{4m^2}\boldsymbol{\sigma} \cdot \nabla V \times \mathbf{p} + \frac{e}{8m^2}\nabla^2 V I_4, \end{aligned} \quad (2.34)$$

where

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}. \quad (2.35)$$

The spin-orbit interaction is described by the sixth term of this Hamiltonian.

2.6 Time-reversal Symmetry

The time-reversal symmetry of the system can be broken down by the nonzero magnetic field. This can be seen as follows. Let us consider the (four-component)

time dependent Schrödinger-Pauli equation:

$$i\hbar \frac{\partial \psi}{\partial t} = H'' \psi, \quad (2.36)$$

where H'' is obtained from the Eq. (2.34) by taking only the terms up to the order $O(1/m)$ into account and by focusing to the particle states and is thus given by

$$H'' = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + eV - \frac{e}{2m} \boldsymbol{\sigma} \cdot \mathbf{B}. \quad (2.37)$$

After reversal of the time to $t \rightarrow -t$ the wave function ψ still satisfies the Eq. (2.36) if the magnetic field \mathbf{B} and the vector potential \mathbf{A} are zero. This can be observed by taking the complex conjugate of Eq. (2.36). However, the nonzero vector potential will cause the term proportional to the $\mathbf{p} \cdot \mathbf{A}$ to the Hamiltonian H'' , which is imaginary and time independent, and the previous discussion does not apply.

3. Theory

In the previous chapter we expressed the conductance through the quantum wire as the trace of the transmission matrix tt^\dagger by using the Landauer formalism. Usually the exact form of transmission matrix is not known. It is nevertheless possible to obtain relevant information about the statistical properties of the transport process by investigating how the eigenvalues of the transmission matrix fluctuate as the function of the L , the length of the wire. That can be done by treating the elements of the transmission matrix as random variables, and by using random matrix theory to evaluate the expectation values of various quantities. It turns out that presence and absence of certain symmetries in the system uniquely determine those expectation values as the function of L . In this chapter, these topics are discussed in greater detail.

3.1 Random matrix theory

Random matrix theory deals with the statistical properties of matrices with randomly distributed elements. The probability distribution of the matrices is taken as input, from which the correlation functions of eigenvalues and eigenvectors are derived as output. For a review, see, e.g, Refs. [11,12]. Wigner-Dyson ensemble [13,14] is an ensemble of $N \times N$ Hermitian matrices \mathcal{H} with the probability distribution of the form

$$P(\mathcal{H}) = C \exp[-\beta \text{Tr} V(\mathcal{H})]. \quad (3.1)$$

With the potential of the form $V(\mathcal{H}) \approx \mathcal{H}^2$, the ensemble is referred as Gaussian. With large N the probability distribution is essentially independent of V . The coefficient β is the number of degrees of freedom in the matrix elements of \mathcal{H} . When $\beta = 1, 2, 4$, the elements of \mathcal{H} are real, complex or quaternion numbers, respectively. A real quaternion q may be represented as a linear combination of unit matrix and Pauli matrices

$$q = \alpha I + ib\sigma_x + ic\sigma_y + id\sigma_z, \quad (3.2)$$

where a, b, c, d are real numbers.

If the transformation $\mathcal{H} \rightarrow U\mathcal{H}U^{-1}$ leaves the probability distribution $P(\mathcal{H})$ invariant, the ensemble of matrices \mathcal{H} obeys a certain symmetry. When U is orthogonal ($\beta = 1$), unitary ($\beta = 2$) or symplectic ($\beta = 4$, a symplectic matrix is a unitary matrix with quaternions as matrix elements), the ensemble is called orthogonal, unitary and symplectic, respectively. Physically the orthogonal ensemble is related to the presence of time-reversal symmetry and the symplectic ensemble to the broken spin-rotation symmetry.

In addition to Wigner-Dyson ensembles one may also consider the so-called Bogoliubov-de Gennes (BdG) [15] and the so called chiral ensembles [16]. The chiral ensembles are applicable in quantum chromodynamics where different ensembles correspond to different choices of the gauge groups and to the number of flavors. We will see in the next chapter, how the BdG ensembles can be used to describe the heat and spin transport in the superconducting quantum wires. The name of these ensembles originates from the fact, that they can be described by a Hamiltonian of Bogoliubov-deGennes type, as was discussed in the previous chapter.

Class	TR	SR	β	α	d	\mathcal{H}	\mathcal{M}
Standard	Yes	Yes	1	1	2	C1	A1
	No	Yes (No)	2	1	2(1)	AIII	A
	Yes	No	4	1	2	DIII	AII
Chiral	Yes	Yes	1	0	2	AI	BDI
	No	Yes (No)	2	0	2(1)	A	AIII
	Yes	No	4	0	2	AII	CII
BdG	Yes	Yes	2	2	4	C	CI
	No	Yes	4	3	4	CII	C
	Yes	No	2	0	2	D	DIII
	No	No	1	0	1	BDI	D

Table 3.1: Classification of symmetry classes according to Cartan’s table [17] The symmetry classes are defined by the presence or absence of time-reversal symmetry (TR) and spin-rotation invariance (SR). The two first indices are symmetry parameters β and α of the random matrix ensembles. The third index d is the multiplicity of the transmission eigenvalues. The two last notations are Cartan’s symbols indicating the symmetric spaces related to the transfer matrix group \mathcal{M} and Hamiltonian \mathcal{H} .

3.2 Classification of symmetry classes of the random matrix theories

In this section we perform the classification of the symmetry classes of the BdG ensembles. The resulting symmetry classes are shown in table (3.1) along with symmetry classes of the Wigner-Dyson and the chiral ensembles.

3.2.1 Symmetry class D

We start by considering systems with the least degree of symmetry, i.e. systems with neither time-reversal symmetry nor spin-rotation invariance. The form (2.13)

of the potential \mathcal{V} ensures that the Hamiltonian \mathcal{H} in Eq. (2.12) obeys particle-hole symmetry, $\mathcal{H} = -\gamma_1 \mathcal{H}^T \gamma_1$ [6]. This implies, that the transfer matrix \mathcal{M} in Eq. (2.14) satisfies $\gamma_1 \mathcal{M} \gamma_1 = \mathcal{M}^*$. In addition, from Eq. (2.14) one finds that the flux conservation (i.e., Hermiticity of \mathcal{H}) implies that $\mathcal{M}^\dagger \tau_3 \mathcal{M} = \tau_3$. The transfer matrix \mathcal{M} obeys the multiplicative rule $\mathcal{M}(z, x) = \mathcal{M}(z, y) \mathcal{M}(y, x)$ for $x < y < z$ and hence is an element of a certain Lie group \mathcal{L} . The appropriate Lie group \mathcal{L} for the symmetry class with neither time-reversal symmetry nor spin-rotation invariance is $\mathcal{L} = O(4N, 4N)$ [6]. By following Ref. [6], we denote the present symmetry class by the symbol D in Cartan's notation.

3.2.2 Symmetry class C

We now consider systems without time-reversal symmetry but with spin-rotation invariance. In that case the Hamiltonian \mathcal{H} obeys [6]

$$\mathcal{H} = -\gamma_2 \mathcal{H}^T \gamma_2. \quad (3.3)$$

This implies that

$$\gamma_2 \mathcal{M} \gamma_2 = \mathcal{M}^*. \quad (3.4)$$

The appropriate Lie group \mathcal{L} for the this symmetry class is $\mathcal{L} = Sp(N, N)$. The symmetry class is denoted by C [6].

3.2.3 Symmetry class DIII

We now consider systems with time-reversal symmetry but without spin-rotation invariance. In that case the Hamiltonian \mathcal{H} obeys [6]

$$\mathcal{H} = \mathcal{T} \mathcal{H}^* \mathcal{T}^{-1}, \quad (3.5)$$

where $\mathcal{T} = i\tau_1 \otimes \sigma_2$. This implies that

$$\mathcal{T}\mathcal{M}\mathcal{T}^{-1} = \mathcal{M}^*. \quad (3.6)$$

The appropriate Lie group \mathcal{L} for the this symmetry class is $\mathcal{L} = O(4N, C)$. The symmetry class is denoted by DIII [6].

3.2.4 Symmetry class CI

We now consider systems with both time-reversal symmetry and spin-rotation invariance. In that case the transfer matrix \mathcal{M} obeys both Eqs. (3.4) and (3.6). The appropriate Lie group \mathcal{L} for the this symmetry class is $\mathcal{L} = Sp(N, C)$. The symmetry class is denoted by CI [6].

3.3 Dorokhov-Mello-Pereyra-Kumar equation

Dorokhov-Mello-Pereyra-Kumar (DMPK) equation is a diffusion-like equation for the probability distribution of transmission eigenvalues. It describes the evolution of eigenvalues as a function of the length of the wire. DMPK equation was derived for standard ensembles ($\alpha = 1$) for $\beta = 1$ by Dorokhov [18] and by Mello, Pereyra and Kumar [19], for $\beta = 2$ by Mello and Stone [1] and for $\beta = 4$ by Macêdo and Chalker [20]. The DMPK equations for chiral and BdG ensembles have been derived by Brouwer et. al. [6, 21].

A given transfer matrix M may be parametrized through the polar decomposition

$$M = \begin{pmatrix} u^{(1)} & 0 \\ 0 & u^{(3)} \end{pmatrix} \begin{pmatrix} \sqrt{I + \Lambda} & \sqrt{\Lambda} \\ \sqrt{\Lambda} & \sqrt{I + \Lambda} \end{pmatrix} \begin{pmatrix} u^{(2)} & 0 \\ 0 & u^{(4)} \end{pmatrix} \equiv UTV. \quad (3.7)$$

Here Λ is a diagonal matrix with nonnegative elements $\lambda_1, \lambda_2, \dots, \lambda_N$. They satisfy

$$T \equiv \text{tr} t t^\dagger = \sum_{\alpha} \frac{1}{1 + \lambda_{\alpha}} \equiv \sum_a \tau_a. \quad (3.8)$$

The physical symmetries of the system can be directly related to the symmetries of the submatrices $u^{(i)}$. For example time-reversal symmetry implies $u^{(3)} = u^{(1)*}, u^{(4)} = u^{(2)*}$.

We know from the elementary probability theory, that probability density function of the sum $Z = X + Y$ of the two statistically independent random variables X and Y is given by convolution of the density functions of X and Y [22], i.e.

$$f_Z(z) = \int f_X(z - y) f_Y(y) dy. \quad (3.9)$$

Therefore, since distinct transfer matrices are supposed to be statistically independent,

$$p_{L+\delta L}(M) = \int p_L(M M_{\delta L}^{-1}) p_{\delta L}(M_{\delta L}) d\tilde{\mu}(M_{\delta L}). \quad (3.10)$$

Here $p_L(M)$ is the probability density for the transfer matrices in a wire segment with length L . The invariant measure $d\tilde{\mu}$ is given by [23]

$$\begin{aligned} d\tilde{\mu}(M) &= J(\lambda) \prod_a d\lambda_a \prod_i d\mu(u^{(i)}), \\ d\mu(u) &= \prod_{a \leq b} \delta s_{ab} \prod_{c < d} \delta a_{cd}, \\ J(\lambda) &= \prod_{a < b} |\lambda_a - \lambda_b|^\beta \prod_c [\lambda_c (1 + \lambda_c)]^{\frac{\alpha-1}{2}}, \end{aligned} \quad (3.11)$$

where, furthermore, we have that $u^\dagger du = \delta a + i\delta s$, where subindices and superscripts have been dropped for clarity. The probability density related to M may thus be obtained by considering the distribution of matrices Λ and the probability density $p_L(\Lambda)$. A small change in the length $\delta L \ll l_{el}$ of the wire may be expected to lead to small changes $\delta \lambda_a \ll 1$ in the parameters λ_a . So the changes $\delta \lambda_a$ may be calculated by using the perturbation theory. Expanding both sides of Eq. (3.10) yields

$$\begin{aligned} p_{L+\delta L}(\Lambda) &= p_L(\Lambda) + \frac{\partial p_L(\Lambda)}{\partial L} \delta L + \dots \\ &= p_L(\Lambda) + \sum_a \frac{\partial p_L(\Lambda)}{\partial \lambda_a} \langle \delta \lambda_a \rangle_{\delta L} + \frac{1}{2} \sum_{ab} \frac{\partial^2 p_L(\Lambda)}{\partial \lambda_a \partial \lambda_b} \langle \delta \lambda_a \delta \lambda_b \rangle_{\delta L} + \dots \end{aligned} \quad (3.12)$$

The probability distribution of the transfer matrices for a short wire segment is assumed to maximize Shannon's information entropy, which is the Gaussian distribution.

By introducing the probability density

$$w_L(\lambda) = p_L(\Lambda) J_\beta(\lambda), \quad (3.13)$$

where

$$J_\beta(\lambda) = \prod_{a < b} |\lambda_a - \lambda_b|^\beta, \quad (3.14)$$

and dimensionless length $s = \frac{L}{Nl}$ we finally obtain

$$\begin{aligned} \frac{\partial w_s(\lambda)}{\partial s} = & \frac{2N}{\beta N + 1 + \alpha - \beta} \sum_{a=1}^N \frac{\partial}{\partial \lambda_a} \left\{ [\lambda_a(1 + \lambda_a)]^{\frac{\alpha+1}{2}} \right. \\ & \left. \times J_\beta(\lambda) \frac{\partial}{\partial \lambda_a} [\lambda_a(1 + \lambda_a)]^{\frac{1-\alpha}{2}} \frac{w_s(\lambda)}{J_\beta(\lambda)} \right\}, \end{aligned} \quad (3.15)$$

the celebrated DMPK equation.

3.4 Conductance cumulants

If the probability distribution function $P(g)$ of the conductance g was known, we would be able to evaluate all the statistical properties of the conductance. However, usually the determination of $P(g)$ is not possible. We can nevertheless gain some information on $P(g)$, if we can evaluate the cumulants of g up to the certain order. Now we give some definitions.

The n :th moment of conductance g is defined by

$$\langle g^n \rangle = \int_{-\infty}^{\infty} g^n P(g) dg. \quad (3.16)$$

The moments are related to the moment generating function

$$Z(t) = \int_{-\infty}^{\infty} \exp(tg) P(g) dg, \quad (3.17)$$

through the derivatives of $Z(t)$:

$$\langle g^n \rangle = \frac{\partial^n}{\partial t^n} Z(t) \Big|_{t=0}. \quad (3.18)$$

The cumulant of order n of g is denoted by $\langle\langle g^n \rangle\rangle$ and may be defined through

$$\ln Z(t) = \sum_{n=1}^{\infty} \frac{\langle\langle g^n \rangle\rangle}{n!} t^n. \quad (3.19)$$

Cumulants are thus related to the cumulant generating function $\varphi(t) = \ln Z(t)$ by

$$\langle\langle g^n \rangle\rangle = \frac{\partial^n \varphi(g)}{\partial t^n} \Big|_{t=0}. \quad (3.20)$$

One thus finds that

$$\begin{aligned} \langle\langle g \rangle\rangle &= \langle g \rangle \\ \langle\langle g^2 \rangle\rangle &= \langle g^2 \rangle - \langle g \rangle^2 \\ \langle\langle g^3 \rangle\rangle &= \langle g^3 \rangle - 3\langle g^2 \rangle \langle g \rangle + 2\langle g \rangle^3 \\ \langle\langle g^4 \rangle\rangle &= \langle g^4 \rangle - 4\langle g^3 \rangle \langle g \rangle - 3\langle g^2 \rangle^2 + 12\langle g^2 \rangle \langle g \rangle^2 - 6\langle g \rangle^4. \end{aligned} \quad (3.21)$$

Let us calculate the cumulants of the normal or Gaussian distribution with expectation value μ and variance σ^2 :

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \quad (3.22)$$

We then find that the cumulant generating function $K(t)$ is now given by

$$K(t) = \mu t + \sigma^2 t^2 / 2. \quad (3.23)$$

The first two cumulants κ_1 and κ_2 are then found to be $\kappa_1 = \mu$, $\kappa_2 = \sigma^2$, while higher order cumulants are equal to zero. Thus non-zero cumulants of the order third or higher indicate a deviation of the distribution from the Gaussian shape.

3.5 Current cumulants

Also the statistics of the current cumulants yields relevant information about the transport process [24]. For the current distribution the random variable is the number of electrons N that have been transmitted through the structure in time t_0 . The

cumulant generating function S is defined by

$$\exp(-S(\chi)) = \sum_N P_{t_0}(N) \exp(iN\chi). \quad (3.24)$$

The current cumulants are defined by

$$C_n = \langle\langle N^n \rangle\rangle = -(-i)^n \frac{\partial^n}{\partial \chi^n} S(\chi) \Big|_{\chi=0}. \quad (3.25)$$

The cumulant generating function of a single channel is [24]

$$S_1(\chi) = (T \exp(\chi) + 1 - T)^{\frac{deVt}{h}}, \quad (3.26)$$

where V is the dc voltage. As the channels are assumed to be independent, for many channels the cumulant generating function has a product form [24]

$$S(\chi) = \prod_j (\tau_j \exp(\chi) + 1 - \tau_j)^{\frac{deVt}{h}}. \quad (3.27)$$

The cumulant generating function is thus given by

$$\ln S(\chi) = \frac{deVt}{h} \sum_j \ln(\tau_j \exp(\chi) + 1 - \tau_j). \quad (3.28)$$

Expanding Eq. (3.28) in terms of χ we find the relation between current cumulants C_k and transmission coefficients T

$$C_k = \frac{deVt}{h} \left\langle \sum_j [T(1 - T) \frac{d}{dT}]^{k-1} T \right\rangle_{T=\tau_j}. \quad (3.29)$$

The first cumulant

$$C_1 = \frac{deVt}{h} \langle \sum_j \tau_j \rangle \quad (3.30)$$

is evidently the charge transported through the structure. The second cumulant

$$C_2 = \frac{deVt}{h} \langle \sum_j (\tau_j - \tau_j^2) \rangle \quad (3.31)$$

is related to the current noise power P by $C_2 = \frac{t}{2e^2} P$.

3.6 Calculation of expectation values

By using the DMPK-equation Eq. (3.15) we obtain the evolution equation for expectation value of any function $F(\lambda)$ of interest as follows:

$$\begin{aligned} \frac{\gamma}{2} \partial_s \langle F \rangle = & \left\langle \sum_a \left(\lambda_a (1 + \lambda_a) \frac{\partial^2 F}{\partial \lambda_a^2} + \left(\frac{1 + \alpha}{2} \right) (1 + 2\lambda_a) \frac{\partial F}{\partial \lambda_a} \right) \right. \\ & \left. + \frac{\beta}{2} \sum_{a \neq b} \frac{\lambda_a (1 + \lambda_a) \frac{\partial F}{\partial \lambda_a} - \lambda_b (1 + \lambda_b) \frac{\partial F}{\partial \lambda_b}}{\lambda_a - \lambda_b} \right\rangle. \end{aligned} \quad (3.32)$$

In order to calculate the first three moments of conductance, we apply Eq. (3.32) to $F = T^p T_q^r$, where

$$T_k = \sum_a \frac{1}{(1 + \lambda_a)^k}. \quad (3.33)$$

A straightforward generalization of the Eq. (B8) of Ref. [1] yields

$$\begin{aligned} \frac{\gamma}{2} \partial_s \langle T^p T_q^r \rangle &= \langle p(p-1) T^{p-2} (T_2 - T_3) T_q^r - 2\nu p T^p T_q^r - \frac{2 - \beta - 2\nu}{2} p T^{p-1} T_2 T_q^r \\ &+ q^2 r(r-1) T^p T_q^{r-2} (T_{2q} - T_{2q+1}) - \frac{\beta}{2} p T^{p+1} T_q^r - (1 + 2\nu - q) r q T^p T_q^r \\ &+ (\nu - q) r q T^p T_q^{r-1} T_{q+1} + 2q r p T^{p-1} T_q^{r-1} (T_{q+1} - T_{q+2}) \\ &+ \frac{\beta}{2} q r T^p T_q^{r-1} \left[\sum_{n=0}^{q-2} (T_{n+1} T_{q-n-1} - T_q) - \sum_{n=0}^{q-1} (T_{n+1} T_{q-n} - T_{q+1}) \right] \rangle, \end{aligned} \quad (3.34)$$

where $\nu = (\alpha - 1)/2$.

3.6.1 Second moment

By choosing $p = 0$, $q = r = 1$ we obtain from Eq. (3.34)

$$\frac{\gamma}{2} \partial_s \langle T \rangle = \langle -2\nu T - \frac{\beta}{2} T^2 + \left(\frac{\beta}{2} - 1 - \nu \right) T_2 \rangle, \quad (3.35)$$

from which the second moment $\langle T^2 \rangle$ can be solved.

3.6.2 Third moment

By choosing $p = 0$, $q = 2$, $r = 1$ in Eq. (3.34) we obtain that

$$\frac{\gamma}{2} \partial_s \langle T_2 \rangle = \langle \beta T^2 - (-2 + \beta + 4\nu) T_2 - 2\beta T T_2 + 2(-2 + \beta + \nu) T_3 \rangle. \quad (3.36)$$

By choosing $p = 2$, $q = r = 0$ in Eq. (3.34) we obtain that

$$\frac{\gamma}{2} \partial_s \langle T^2 \rangle = \langle -\beta T^3 - 4\nu T^2 + 2T_2 + (-2 + \beta + 2\nu) T T_2 - 2T_3 \rangle. \quad (3.37)$$

The third moment $\langle T^3 \rangle$ can be solved from Eqs. (3.36) and (3.37).

3.7 Alternative form of the DMPK equation

The equation (3.15) can be also written in the form of the Fokker-Planck equation

$$\frac{\partial w_s(\lambda)}{\partial s} = \sum_{a=1}^N \frac{\partial}{\partial \lambda_a} D(\lambda_a) \left(\frac{\partial w_s(\lambda)}{\partial \lambda_a} + w_s(\lambda) \frac{\partial}{\partial \lambda_a} \Omega(\lambda) \right), \quad (3.38)$$

where

$$D(\lambda) = \frac{2}{\gamma} \lambda (1 + \lambda), \quad (3.39)$$

$$\Omega(\lambda) = -\ln J(\lambda), \quad (3.40)$$

and $\gamma = \beta + \frac{1+\alpha-\beta}{N}$. In the previous equation, we want to transform the variables so that we get rid of the λ -dependence of the diffusion coefficient D . This is accomplished by

$$\lambda_a = \sinh^2 q_a. \quad (3.41)$$

Therefore

$$\frac{\partial w_s(q)}{\partial s} = \frac{1}{2\gamma} \sum_{i=1}^N \frac{\partial}{\partial q_i} J(q) \frac{\partial}{\partial q_i} \frac{w_s(q)}{J(q)}. \quad (3.42)$$

4. Calculations

4.1 Calculation of the correlation function

Our starting point is the Fokker-Planck equation (3.42) [25] in the limit $N \gg 1$ when $\gamma \approx \beta$:

$$\frac{\partial P}{\partial t} = \frac{1}{2\beta} \sum_{i=1}^N \frac{\partial}{\partial q_i} J \frac{\partial P}{\partial q_i} \frac{1}{J}, \quad (4.1)$$

where

$$J = \prod_{i < j} |\cosh(2q_i) - \cosh(2q_j)|^\beta \prod_{i=1}^N \sinh^\alpha(2q_i), \quad (4.2)$$

and where we have denoted the probability density w_s in Eq. (3.42) by P . After changing to new variables $x_i = \cosh(2q_i)$ we find that the Eq.(4.1) takes the following form [26]

$$\frac{\partial P}{\partial t} = \mathcal{L}_{\text{FP}} P. \quad (4.3)$$

where

$$\mathcal{L}_{\text{FP}} = \frac{2}{\beta} \sum_{i=1}^N \frac{\partial}{\partial x_i} J_\beta w_N s(x_i) \frac{\partial}{\partial x_i} \frac{1}{w_N J_\beta}. \quad (4.4)$$

with

$$J_\beta(\{x\}) = |\Delta_N(\{x\})|^\beta, \quad \Delta_N(\{x\}) = \prod_{i < j} (x_i - x_j), \quad w_N(\{x\}) = \prod_{i=1}^N w(x_i), \quad (4.5)$$

and

$$w(x) = (x^2 - 1)^{(\alpha-1)/2}, \quad s(x) = x^2 - 1. \quad (4.6)$$

In the following, we focus on the case $\beta = 2$.

The general solution of the Fokker-Planck equation (4.3) can be written as

$$P(\{x\}, t) = \int d^N x' \mathcal{P}(\{x\}, t | \{x'\}, 0) P(\{x'\}, 0), \quad (4.7)$$

where the transition probability satisfies the Fokker-Planck equation

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\text{FP}} \right) \mathcal{P}(\{x\}, t | \{x'\}, 0) = 0 \quad (4.8)$$

with the ballistic initial condition

$$P(\{x\}, 0) = \prod_{i=1}^N \delta(x_i - 1). \quad (4.9)$$

One of the methods to solve Fokker-Planck equation (4.8) is based on mapping the non-Hermitian operator \mathcal{L}_{FP} onto a Hermitian operator \mathcal{H} , which can be interpreted as a Hamiltonian of a quantum system [26]. \mathcal{H} can be expressed as the sum of single-particle Hamiltonians $\mathcal{H} = \sum_{i=1}^N H_{x_i}$, where

$$H_x = -\frac{1}{w(x)} \frac{d}{dx} \left(w(x) s(x) \frac{d}{dx} \right) = -s(x) \frac{d^2}{dx^2} - r(x) \frac{d}{dx}, \quad (4.10)$$

and where $r(x) = (1 + \alpha)x$. Furthermore, we want to express the transition probability in Eq. (4.7) in a determinantal form

$$\mathcal{P}(\{x\}, t | \{x'\}, 0) = C_N \det[K(x_i, x_j; t)]_{i,j=1,\dots,N}, \quad (4.11)$$

with $C_N = 1/N!$ and with a kernel $K(x, y; t)$ satisfying the properties

$$\int_a^b dx K(x, x; t) = N, \quad (4.12)$$

$$\int_a^b dz K(x, z; t) K(z, y; t) = K(x, y; t). \quad (4.13)$$

This is accomplished by using the biorthogonal functions method [26], which yields that the kernel $K(x, y; t)$ can be written in a product form

$$K(x, y; t) = w(x) \sum_{n=0}^{N-1} \psi_n(x, t) \chi_n(y, t), \quad (4.14)$$

where

$$\chi_m(x, t) = g(x, t|x'_{m+1}, 0), \quad m = 0, 1, \dots, N-1. \quad (4.15)$$

The one-particle Green's function $g(x, t|x'_{m+1}, 0)$ has the following spectral decomposition

$$g(x, t|y, t') = \int_0^\infty dk e^{-\varepsilon_k(t-t')} \varphi_k(x) \varphi_k(y), \quad (4.16)$$

where eigenfunctions $\varphi_k(x)$ satisfy the eigenvalue equation $H_x \varphi_k(x) = \varepsilon_k \varphi_k(x)$, with continuum spectra

$$\varepsilon_k = k^2 + \alpha^2/4, \quad k \geq 0, \quad (4.17)$$

and are given in terms of the hypergeometric function

$$\varphi_k(x) = A_k \mathcal{F}_k^{(\alpha)}(x); \quad \mathcal{F}_k^{(\alpha)}(x) = F\left(\frac{\alpha}{2} + ik, \frac{\alpha}{2} - ik; \frac{1+\alpha}{2}; \frac{1-x}{2}\right) \quad (4.18)$$

with

$$A_k = \frac{2^{(1-\alpha)/2} |\Gamma(\alpha/2 + ik)|}{\Gamma(\alpha/2 + 1/2) |\Gamma(ik)|}. \quad (4.19)$$

Likewise,

$$\psi_n(x, t) = \int_{-1}^1 dy w(y) L_{n+1}(y) g_2(x, t|y, 0), \quad n = 0, 1, \dots, N-1, \quad (4.20)$$

where the Green's function $g_2(x, t|y, 0)$ is given by

$$g_2(x, t|y, 0) = \sum_{n=0}^{\infty} e^{-\varepsilon_n t} \phi_n(x) \phi_n(y). \quad (4.21)$$

Eigenfunctions ϕ_n satisfy the eigenvalue equation $H_x \phi_n = -\epsilon_n \phi_n$, with a discrete spectrum

$$\epsilon_n = (n + \alpha), \quad n = 0, 1, 2, \dots, \quad (4.22)$$

and are normalized Jacobi polynomials

$$\phi_n(x) = \frac{1}{\sqrt{h_n}} P_n^{\nu, \nu}(x), \quad h_n = \frac{2^{2\nu+1} (\Gamma(n + \nu + 1))^2}{n! (2n + 2\nu + 1) \Gamma(n + 2\nu + 1)}, \quad (4.23)$$

with parameter $\nu = (\alpha - 1)/2$. Finally, $L_{n+1}(x)$ is the Lagrange interpolation polynomial defined by

$$L_{n+1}(x) = \prod_{l=0, l \neq n}^{N-1} \frac{x - x'_{l+1}}{x'_{n+1} - x'_{l+1}}, \quad n = 0, \dots, N-1. \quad (4.24)$$

One can show [26] that the functions $\psi_n(x, t)$ and $\chi_n(x, t)$ satisfy the biorthogonality condition

$$\int_1^\infty dx w(x) \psi_n(x, t) \chi_m(x, t) = \delta_{n,m}, \quad (4.25)$$

from which the properties (4.12) and (4.13) of the kernel $K(x, y; t)$ follow immediately.

By using the Eqs. (4.14), (4.15) and (4.20) and Cauchy integral formula we find that

$$K(x, y; t) = w(x) \int_{-1}^1 d\xi v(\xi) \oint \frac{dz}{2\pi i} \frac{1 - \Omega(\xi, z)}{(z - \xi)} g(y, t|z, 0) g_2(x, t|\xi, 0), \quad (4.26)$$

where

$$\Omega(\xi, z) \equiv \prod_{l=1}^N \frac{\xi - x'_l}{z - x'_l} = 1 - (z - \xi) \sum_{n=1}^N \frac{L_n(\xi)}{z - x'_n} \quad (4.27)$$

and where $v(x) = w(x)$. By using Eqs. (4.16), (4.18), (4.21) and (4.23) we finally find [26]

$$\begin{aligned} K(x, y; s) = & \frac{2^{1-\alpha}(x^2 - 1)^{(\alpha-1)/2}}{\Gamma(\alpha/2 + 1/2)} \sum_{n=0}^{N-1} (-1)^n \frac{(2n + \alpha)\Gamma(n + \alpha)}{\Gamma(n + \alpha/2 + 1/2)} P_n^{(\nu, \nu)}(x) \\ & \times \int_0^\infty \frac{dk}{\pi} \frac{k \sinh(k\pi)}{\varepsilon_{nk}} c_{nk}^{(N)} e^{-\varepsilon_{nk}s} F_k^{(\nu)}(y), \end{aligned} \quad (4.28)$$

where $\varepsilon_{nk} = k^2 + (n + \nu + 1/2)^2$ and $\nu = (\alpha - 1)/2$. Finally,

$$c_{nk}^{(N)} = \frac{|\Gamma(N + \nu + 1/2 + ik)|^2}{(N - n - 1)! \Gamma(N + n + 2\nu + 1)}. \quad (4.29)$$

4.2 Three first moments of the conductance

4.2.1 First Moment

In this section, we calculate the first three moments of the heat conductance in dirty superconducting wires with the presence of the time reversal symmetry ($\beta = 2$), mostly following the Ref. [25]. Unlike in Ref. [25], we fix the parameter α already at the beginning of the calculation to the value $\alpha = 2$, which means according the Table

(3.1), that we are concentrating to the system, which belongs to the Bogoliubov-de Gennes universality class with the presence of the spin-rotation invariance. We take also the limit $N \rightarrow \infty$, where N is the number of open scattering channels.

The dimensionless heat conductance (in units of G_0) is given by the Landauer formula (see Eq. (2.10))

$$g = \sum_{i=1}^N \tau_i = \sum_{i=1}^N \frac{2}{1 + x_i}, \quad (4.30)$$

where $\tau_i = 1/\cosh^2(q_i)$ is the transmission eigenvalue of the i :th scattering channel and $x_i = \cosh(2q_i)$.

By definition, the n -th integer moment of conductance is

$$\langle g^n \rangle = \sum_{i_1, \dots, i_N=1}^N \int_1^\infty dx_{i_1} \dots \int_1^\infty dx_{i_N} 2^N \frac{1}{1 + x_{i_1}} \dots \frac{1}{1 + x_{i_N}} P(x_{i_1}, \dots, x_{i_N}; s), \quad (4.31)$$

where the probability distribution $P(\{x\}, s)$ is given in the Eq. (4.7). In particular, the first and second moments are given by

$$\langle g \rangle = \int_1^\infty dx \frac{2R(x; s)}{1 + x}, \quad (4.32)$$

$$\langle g^2 \rangle = \int_1^\infty dx \int_1^\infty dy \frac{4R(x, y; s)}{(1 + x)(1 + y)} + \int_1^\infty dx \frac{4R(x; s)}{(1 + x)^2}, \quad (4.33)$$

where the n -point correlation function, $1 \leq n \leq N$, is given by

$$R(x_1, \dots, x_N; s) = \frac{N!}{(N - n)!} \int_1^\infty dx_{n+1} \dots \int_1^\infty dx_N P(x_1, \dots, x_N; s). \quad (4.34)$$

The combinatorial factor $N!/(N - n)!$ appears since the probability distribution $P(x_1, \dots, x_N; s)$ is invariant under any permutation of x_1, \dots, x_N . From Eqs. (4.11)-(4.13) it follows that

$$R(x_1, \dots, x_n; s) = \det(K(x_i, x_j; s))_{i,j=1, \dots, n}, \quad (4.35)$$

where the function $K(x_i, x_j; s)$ is given by Eq. (4.28) ([25], [26])

$$\begin{aligned} K(x, y; s) &= \frac{2^{1-\alpha}(x^2 - 1)^{(\alpha-1)/2}}{\Gamma(\alpha/2 + 1/2)} \sum_{n=0}^{N-1} (-1)^n \frac{(2n + \alpha)\Gamma(n + \alpha)}{\Gamma(n + \alpha/2 + 1/2)} P_n^{(\nu, \nu)}(x) \\ &\times \int_0^\infty \frac{dk}{\pi} \frac{k \sinh(k\pi)}{\varepsilon_{nk}} c_{nk}^{(N)} e^{-\varepsilon_{nk}s} F_k^{(\nu)}(y). \end{aligned} \quad (4.36)$$

By using Eq. (8.325) of Ref. [27]:

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)} = \prod_{k=0}^{\infty} \left[\left(1 + \frac{\gamma}{\alpha+k}\right) \left(1 - \frac{\gamma}{\beta+k}\right) \right], \quad (4.37)$$

and the identity $\overline{\Gamma(z)} = \Gamma(\bar{z})$, we find that

$$c_{nk}^{(N)} = \prod_{j=0}^{\infty} \left(\frac{(N+2+n+j)(N+j-n)}{(N+1+j)^2 + k^2} \right) \rightarrow 1, \quad (4.38)$$

when $N \rightarrow \infty$ and $\nu = 1/2$. By using Eq. (15.1.16) of Ref. [28]:

$$F(a, 2-a; \frac{3}{2}; \sin^2(z)) = \frac{\sin[(2a-2)z]}{(a-1)\sin(2z)} \quad (4.39)$$

we find

$$F_k^{(1/2)}(\cosh(2t)) = \frac{\sin(2kt)}{k \sinh(2t)}. \quad (4.40)$$

We thus find, that

$$\begin{aligned} K(\cosh(2t), \cosh(2t); s) &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)\Gamma(n+2)}{\Gamma(n+3/2)} P_n^{(1/2, 1/2)}(\cosh(2t)) \\ &\times \int_0^{\infty} \frac{dk}{\pi} \frac{\sinh(k\pi)}{\varepsilon_{nk}} e^{-\varepsilon_{nk}s} |\sin(2kt)| \end{aligned} \quad (4.41)$$

and

$$\langle g \rangle = 2 \int_{-\infty}^{\infty} dt |\tanh(t)| K(\cosh(2t), \cosh(2t); s), \quad (4.42)$$

where we have used the evenness of the integrand to extend the semi-infinite integration region to infinite. The equation (4.42) can thus be interpreted as $-i$ times the Fourier transform of the function involving Jacobi polynomials.

We evaluate the integral (4.42) by following the scheme presented in the Ref. [29]. Let us consider the following integral:

$$\begin{aligned} I_{nk}^{\gamma, \delta}(\alpha, \beta) &\equiv \int_{-\infty}^{\infty} e^{-2ixk} (1 - \tanh(x))^{\alpha} (1 + \tanh(x))^{\beta} P_n^{\gamma, \delta}(\cosh(2x)) dx \\ &= \int_{-1}^1 (1-t)^{\alpha-1+ik} (1+t)^{\beta-1-ik} P_n^{\gamma, \delta} \left(\frac{2}{(1-t)(1+t)} - 1 \right) dt \\ &= 2^{\alpha+\beta-1} \int_0^1 u^{\alpha-1+ik} (1-u)^{\beta-1-ik} P_n^{\gamma, \delta} \left(\frac{1}{2(u)(1-u)} - 1 \right) du, \end{aligned} \quad (4.43)$$

where we have made the substitutions $t = \tanh(x)$, $t = 1 - 2u$ and used the identities $\frac{dt}{dx} = (\cosh(x))^{-2} = (1 - \tanh(x))(1 + \tanh(x))$, $\frac{1 - \tanh(x)}{1 + \tanh(x)} = e^{-2x}$. By using the following expression

$$P_n^{\gamma, \delta}(x) = \frac{(\gamma + 1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k (n + \gamma + \delta + 1)_k}{(\gamma + 1)_k} \frac{[(1 - x)/2]^k}{k!} \quad (4.44)$$

for the Jacobi polynomial, where the Pochhammer's symbol $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$, we obtain that

$$\begin{aligned} P_n^{\gamma, \delta} \left(\frac{1}{2(u)(1-u)} - 1 \right) &= \frac{(\gamma + 1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k (n + \gamma + \delta + 1)_k}{k! (\gamma + 1)_k} \\ &\quad \times \sum_{l=0}^k \binom{k}{l} \frac{1}{(-4)^l u^l (1-u)^l} \\ &= \frac{(\gamma + 1)_n}{n!} \sum_{l=0}^n \frac{1}{(-4)^l u^l (1-u)^l} \\ &\quad \times \sum_{k=l}^n \binom{k}{l} \frac{(-n)_k (n + \gamma + \delta + 1)_k}{k! (\gamma + 1)_k}. \end{aligned} \quad (4.45)$$

By using the following expression

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (4.46)$$

for the hypergeometric function, we find that

$$\begin{aligned} P_n^{\gamma, \delta} \left(\frac{1}{2(u)(1-u)} - 1 \right) &= \frac{(\gamma + 1)_n}{n!} \sum_{l=0}^n \frac{1}{(-4)^l u^l (1-u)^l} \\ &\quad \times \frac{\Gamma(l - n) \Gamma(\gamma + 1) \Gamma(l + n + \gamma + \delta + 1)}{l! \Gamma(l + \gamma + 1) \Gamma(-n) \Gamma(n + \gamma + \delta + 1)} \\ &\quad \times F(l - n, l + n + \gamma + \delta + 1; l + \gamma + 1; 1). \end{aligned} \quad (4.47)$$

By using the Gauss's theorem

$$F(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}, \quad (4.48)$$

we find that

$$P_n^{\gamma, \delta} \left(\frac{1}{2(u)(1-u)} - 1 \right) = \frac{(\gamma + 1)_n}{n!} \sum_{l=0}^n \frac{1}{(-4)^l u^l (1-u)^l} f_n^{\gamma, \delta}(l), \quad (4.49)$$

where

$$\begin{aligned} f_n^{\gamma,\delta}(l) &= \frac{\Gamma(l-n)\Gamma(\gamma+1)\Gamma(l+n+\gamma+\delta+1)}{l!\Gamma(-n)\Gamma(n+\gamma+\delta+1)} \frac{\Gamma(-l-\gamma)}{\Gamma(n+\gamma+1)\Gamma(-n-\gamma)} \\ &= \frac{(-n)_l(n+\gamma+\delta+1)_l\Gamma(-l-\gamma)}{l!(\gamma+1)_n\Gamma(-n-\gamma)}. \end{aligned} \quad (4.50)$$

By definition, the Euler beta function is

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \quad (4.51)$$

We thus find from the Eqs. (4.43) and (4.49) that

$$\begin{aligned} I_{nk}^{\gamma,\delta}(\alpha, \beta) &= 2^{\alpha+\beta-1} \frac{(\gamma+1)_n}{n!} \sum_{l=0}^n \frac{f_n^{\gamma,\delta}(l)}{(-4)^l} B(\alpha-l+ik, \beta-l-ik) \\ &= 2^{\alpha+\beta-1} \frac{(\gamma+1)_n}{n!} \sum_{l=0}^n \frac{f_n^{\gamma,\delta}(l)}{(-4)^l} \frac{\Gamma(\alpha-l+ik)\Gamma(\beta-l-ik)}{\Gamma(\alpha+\beta-2l)}. \end{aligned} \quad (4.52)$$

It can be seen in the Eq. (4.52), that due to the Gamma function in the denominator, the sum vanishes when $2l \geq \alpha + \beta$. Thus we find, that e.g. $I_{nk}^{\gamma,\delta}(0, 0) = 0$.

It now follows from the Eqs. (4.42), (4.41) and (4.43) that

$$\begin{aligned} \langle g \rangle &= \frac{4}{i\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(n+1)^2 n!}{\Gamma(n+3/2)} \\ &\quad \times \int_0^{\infty} \frac{dk \sinh(k\pi)}{\pi \varepsilon_{nk}} e^{-\varepsilon_{nk}s} I_{nk}^{1/2,1/2}(0, 1), \end{aligned} \quad (4.53)$$

where we have changed the order of the x - and k -integrations. By using Eqs. (4.52) and (4.50) we can see that Eq. (4.53) equals to

$$\langle g \rangle = 8 \sum_{n=0}^{\infty} (n+1)^2 \int_0^{\infty} \frac{dk}{\pi} \frac{e^{-\varepsilon_{nk}s}}{\varepsilon_{nk}}. \quad (4.54)$$

It can be easily seen, that when $\nu = 1/2$ and $N \rightarrow \infty$, this corresponds Eq. (6) for $\langle g^1 \rangle$ in Ref. [25].

4.2.2 Second Moment

We recall that by definition, the second moment of the conductance is given by

$$\langle g^2 \rangle = \int_1^{\infty} dx \int_1^{\infty} dy \frac{4R(x, y; s)}{(1+x)(1+y)} + \int_1^{\infty} dx \frac{4R(x; s)}{(1+x)^2}. \quad (4.55)$$

By using the expression (4.35) for the 2-point correlation function, we find that

$$R(x, y; s) = K(x, x; s)K(y, y; s) - K(x, y; s)K(y, x; s). \quad (4.56)$$

Let us consider first the second term of the Eq. (4.55). By using Eqs. (4.41) and (4.43) we find that

$$\begin{aligned} \int_1^\infty dx \frac{4R(x; s)}{(1+x)^2} &= \frac{4}{i\sqrt{\pi}} \sum_{n=0}^\infty (-1)^{n+1} \frac{(n+1)^2 n!}{\Gamma(n+3/2)} \\ &\times \int_0^\infty \frac{dk}{\pi} \frac{\sinh(k\pi)}{\varepsilon_{nk}} e^{-\varepsilon_{nk}s} \left(I_{nk}^{1/2, 1/2}(1, 2) - I_{nk}^{1/2, 1/2}(1, 1) \right) \\ &= 8 \sum_{n=0}^\infty (n+1)^2 \int_0^\infty \frac{dk}{\pi} g_{(2)nk} \frac{e^{-\varepsilon_{nk}s}}{\varepsilon_{nk}}, \end{aligned} \quad (4.57)$$

where

$$g_{(2)nk} = 2k^2 - 2n(n+2)/3. \quad (4.58)$$

Let us then consider the first term of the Eq. (4.55). The contribution of the first term of the Eq. (4.56) to that is

$$64 \sum_{n, n'=0}^\infty (n+1)^2 (n'+1)^2 \int_0^\infty \frac{dk}{\pi} \int_0^\infty \frac{dk'}{\pi} \frac{e^{-[(\varepsilon_{nk} + \varepsilon_{n'k'})s]}}{\varepsilon_{nk} \varepsilon_{n'k'}}, \quad (4.59)$$

which, by interchanging k and k' , (if necessary), can be seen to be equal to the contribution of the second term of the Eq. (4.56). Hence their difference seems to be zero! Hence it seems, that $\langle g^2 \rangle$ is given by the rhs. of the Eqs. (4.57) and (4.58). However, in Ref. [25] the expression for $\langle g^2 \rangle$ was obtained, that corresponds to the our result (4.57) if $g_{(2)nk}$ is replaced by

$$g_{nk}^{(2)} = 2k^2 + 2n(n+2)/3. \quad (4.60)$$

We can nevertheless obtain the previous identity by using the Eq. (3.35), which gives for $\gamma = \beta = 2$ and $\nu = 1/2$

$$g_{nk}^{(2)} = ((n+1)^2 + k^2 - 1)g_{nk}^{(1)} + \frac{1}{2}g_{(2)nk} = 2k^2 + 2n(n+2)/3. \quad (4.61)$$

4.2.3 Third moment

From now on T_m given by Eq. (3.33) is denoted by g_m . By proceeding in a similar manner than before we obtain that

$$\langle g_m \rangle = \int_1^\infty dx \frac{2^m R(x; s)}{(1+x)^m} = 2 \int_{-\infty}^\infty dt \frac{\tanh(t)}{\cosh^2(t)^{m-1}} K(\cosh(2t), \cosh(2t); s). \quad (4.62)$$

By using Eqs. (4.41) and (4.43) and we get that

$$\begin{aligned} \langle g_m \rangle &= \frac{4}{i\sqrt{\pi}} \sum_{n=0}^\infty (-1)^{n+1} \frac{(n+1)^2 n!}{\Gamma(n+3/2)} \\ &\times \int_0^\infty \frac{dk}{\pi} \frac{\sinh(k\pi)}{\varepsilon_{nk}} e^{-\varepsilon_{nk}s} \left(I_{nk}^{1/2,1/2}(m-1, m) - I_{nk}^{1/2,1/2}(m-1, m-1) \right). \end{aligned} \quad (4.63)$$

Hence we obtain that

$$\langle g_3 \rangle = 8 \sum_{n=0}^\infty (n+1)^2 \int_0^\infty \frac{dk}{\pi} g_{(3)nk} \frac{e^{-\varepsilon_{nk}s}}{\varepsilon_{nk}}, \quad (4.64)$$

where

$$g_{(3)nk} = \frac{2}{15} (5k^4 + 5k^2(1 - 2n(n+2)) + (n-1)n(n+2)(n+3)). \quad (4.65)$$

In Ref. [25] the value for third moment $\langle g^3 \rangle$ was given by Eq. (4.64) if $g_{(3)nk}$ is replaced by

$$g_{nk}^{(3)} = \frac{2}{15} (5k^4 + 5k^2(1 + 4n(n+2)) + (n-1)n(n+2)(n+3)). \quad (4.66)$$

We can nevertheless obtain the previous identity by using the Eqs. (3.36) and (3.37), which give for $\gamma = \beta = 2$ and $\nu = 1/2$

$$\begin{aligned} g_{nk}^{(3)} &= \frac{1}{8} [((n+1)^2 + k^2 + 6)g_{(2)nk} + (4(n+1)^2 + 4k^2 - 6)g_{nk}^{(2)} - 7g_{(3)nk}] \\ &= \frac{2}{15} (5k^4 + 5k^2(1 + 4n(n+2)) + (n-1)n(n+2)(n+3)). \end{aligned} \quad (4.67)$$

4.3 Moments in Different Regimes

4.3.1 Metallic Regime

In the metallic regime $s \ll 1$. Eq. (4.54) can be easily integrated and we obtain

$$\langle g \rangle = 4 \sum_{n=0}^\infty (n+1) \operatorname{erfc}((n+1)\sqrt{s}), \quad (4.68)$$

where $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$ is the complementary error function. Rather than \sqrt{s} , we want use $1/s$ as the expansion parameter. The progress can be made by using the Poisson summation formula, which says that

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \quad (4.69)$$

for sufficiently smooth function f , where $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx$ is the Fourier transform of f . If we set $f(x) = 8x \text{erfc}(x\sqrt{s})$, we find that

$$\hat{f}(n) = -\frac{2}{n^2 \pi^2} + 2 \left(\frac{2}{s} + \frac{1}{n^2 \pi^2} \right) e^{-n^2 \pi^2 / s}. \quad (4.70)$$

Therefore

$$\langle g \rangle = \sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \hat{f}(n) + \hat{f}(0)/2. \quad (4.71)$$

By Taylor expanding the exponential in $\hat{f}(n)$ we obtain

$$\hat{f}(0) = \frac{2}{s} \quad (4.72)$$

and hence

$$\langle g \rangle = \frac{1}{s} - \frac{1}{3} + 2 \sum_{n=1}^{\infty} \left(\frac{2}{s} + \frac{1}{n^2 \pi^2} \right) e^{-n^2 \pi^2 / s}. \quad (4.73)$$

The first two terms of Eq. (4.73) correctly reproduce the weak localization correction to the Ohm's law shown in the Eq. (2.15). The remaining terms have the nonanalytic dependence on the wire length.

For the variance of the conductance, we obtain similarly from Eqs. (4.57) and (4.60) that

$$\text{var}(g) = \langle g^2 \rangle - \langle g \rangle^2 = \frac{1}{15} + 4 \left(-\frac{2\pi^2}{3s^3} - \frac{1}{s^2} + \frac{1}{\pi^4} \right) e^{-\pi^2 / s}, \quad (4.74)$$

where we have taken into the account only the terms up to the first order in n . The first term in Eq. (4.74) correctly reproduces the universal conductance fluctuations shown in the Eq. (2.25).

For the third conductance cumulant, we obtain that

$$\langle g^3 \rangle_c = 4A_s^{(2)} e^{-\pi^2 / s}, \quad (4.75)$$

where $A_s^{(2)} = 2\pi^4/(15s^5) + \pi^2/(3s^4) - 2/(3s^3) - 5/(\pi^2s^2) - 10/(\pi^4s) - 7/\pi^6$. It can be observed, that third cumulant has only terms which depend nonanalytically on the wire length.

4.3.2 Localized Regime

In the limit $s \gg 1$ we get from the Eq. (4.68) that

$$\langle g \rangle = 4\text{erfc}(\sqrt{s}) \approx 4 \frac{e^{-s}}{\sqrt{\pi s}}, \quad (4.76)$$

where the approximation $\text{erfc}(x) \approx \frac{e^{-x^2}}{x\sqrt{\pi}}$ is used.

Similarly one obtains that

$$\text{var}(g) = 3\langle g^3 \rangle_c = 4 \frac{e^{-s}}{\sqrt{\pi s^3}}, \quad (4.77)$$

where $\langle g^3 \rangle_c$ denotes the third cumulant of the conductance.

4.4 Current cumulants in metallic regime

We can compute the current cumulants in metallic regime in any finite order by using Eqs. (4.63) and (3.29). We propose the following ansatz for $\langle g_m \rangle$:

$$\langle g_m \rangle = \sum_{i_1=0}^{m-1} \sum_{i_2=0}^{m-1} \sum_{n=1}^{\infty} \int_0^{\infty} dk c_{i_1 i_2}^m f(k, n) k^{2i_1} n^{2i_2+2}, \quad (4.78)$$

where

$$f(k, n) = \frac{e^{-\epsilon_{nk}s}}{\pi \epsilon_{nk}}, \quad (4.79)$$

where

$$\epsilon_{nk} = k^2 + n^2. \quad (4.80)$$

Performing the k -integration yields

$$\langle g_m \rangle = \sum_{i_1=0}^{m-1} \sum_{i_2=0}^{m-1} \sum_{n=1}^{\infty} c_{i_1 i_2}^m f_{i_1}(n) n^{2i_2+2}, \quad (4.81)$$

where

$$f_{i_1}(n) = \frac{s^{1/2-i_1}}{2\pi} E_{1/2+i_1}(n^2 s) \Gamma(1/2 + i_1), \quad (4.82)$$

where $E_{1/2+i_1}(n^2 s)$ is the exponential integral which is defined by the formula

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt. \quad (4.83)$$

The Fourier transform of $f_{i_1}(n)$ is given by

$$\hat{f}_{i_1}(n) = (2\pi n)^{-2i_1} ((2i_1 - 1)! - \frac{4^{i_1}}{2\sqrt{\pi}} \Gamma(\frac{1}{2} + i_1) \Gamma(i_1, \frac{n^2 \pi^2}{s})), \quad (4.84)$$

where $\Gamma(i_1, \frac{n^2 \pi^2}{s})$ is the incomplete gamma function which is defined by the formula

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt. \quad (4.85)$$

In the limit $s \ll 1$ the $\langle g_m \rangle$ is then given by

$$\langle g_m \rangle = \sum_{n=1}^\infty (S_m^1(n) + S_m^2(n) + S_m^3(n)) + \frac{1}{2} (S_m^1(0) + S_m^2(0) + S_m^3(0)), \quad (4.86)$$

where

$$S_m^1(n) = - \sum_{i_2=1}^m c_{0i_2}^m (-1)^{i_2} (2\pi)^{-2i_2} \frac{\partial^{2i_2-1}}{\partial n^{2i_2-1}} \frac{1 - e^{-n^2 \pi^2 / s}}{n}, \quad (4.87)$$

$$S_m^2(n) = \sum_{i_1=1}^{m-1} \sum_{i_2=1}^m c_{i_1 i_2}^m (-1)^{i_2} (2\pi)^{-2i_2} (2i_1 + 2i_2 - 1)! (2\pi n)^{-(2i_1+2i_2)} \quad (4.88)$$

and

$$S_m^3(n) = \sum_{i_1=1}^{m-1} \sum_{i_2=1}^m \sum_{j=i_2}^{2i_2} \frac{4^{i_1}}{2\sqrt{\pi}} c_{i_1 i_2}^m (-1)^{i_2+j} (2\pi)^{-2i_2} \times C_{ji_2} (2\pi n)^{-(2i_1+2i_2)} \Gamma(i_1 + j, \frac{n^2 \pi^2}{s}), \quad (4.89)$$

where $C_{ji_2} = \frac{(2j)!}{j!} \delta_{ji_2}$ is the coefficient of the term $x^{2j-2i_2} \frac{d^j}{dx^j} f(x^2)$ of the expression $\frac{d^{2j}}{dx^{2j}} f(x^2)$ where f is arbitrary sufficiently smooth function. In the derivation of Eqs. (4.86)-(4.89) we have used the Eq. 6.5.26 of Ref. [28]:

$$\frac{\partial^n}{\partial x^n} [x^{-a} \Gamma(a, x)] = (-1)^n x^{-a-n} \Gamma(a + n, x), \quad (4.90)$$

where n is the positive integer. In addition, we have used the fact that

$$f_0(n) = \frac{\operatorname{erfc}(n\sqrt{s})}{2n} \quad (4.91)$$

along the Fourier sine transform of the complementary error function:

$$\int_0^\infty \operatorname{erfc}(x) \sin(xy) dx = \frac{1 - e^{-\frac{y^2}{4}}}{y}. \quad (4.92)$$

5. Results

By using the fact, that in our notation $\langle \sum_j \tau_j^k \rangle = \langle g_k \rangle$, Eqs. (4.86)-(4.89) and Eq. (3.29) yield the following current cumulants up to the tenth order:

$$\begin{aligned}
\Psi_0 &= \frac{deVt}{h} \frac{1}{s} \\
\Psi_1 &= -\frac{1}{3} \frac{deVt}{h} \\
C_1 &= \Psi_0 + \Psi_1 \\
C_2 &= \frac{1}{3} \Psi_0 + \frac{1}{15} \Psi_1 \\
C_3 &= \frac{1}{15} \Psi_0 + \frac{1}{315} \Psi_1 \\
C_4 &= -\frac{1}{105} \Psi_0 - \frac{11}{1575} \Psi_1 \\
C_5 &= -\frac{1}{105} \Psi_0 - \frac{1}{1485} \Psi_1 \\
C_6 &= \frac{1}{231} \Psi_0 + \frac{47221}{14189175} \Psi_1 \\
C_7 &= \frac{27}{5005} \Psi_0 + \frac{811}{2027025} \Psi_1 \\
C_8 &= -\frac{3}{715} \Psi_0 - \frac{1790851}{516891375} \Psi_1 \\
C_9 &= -\frac{233}{36465} \Psi_0 - \frac{98299813}{206239658625} \Psi_1 \\
C_{10} &= \frac{6823}{969969} \Psi_0 + \frac{23610799591}{3781060408125} \Psi_1
\end{aligned} \tag{5.1}$$

whereas

$$\begin{aligned}
C_{11} &= \frac{62003}{4849845}\Psi_0 + \frac{266244989}{279030126375}\Psi_1 \\
C_{12} &= -\frac{184337}{10140585}\Psi_0 - \frac{111748334272999}{6474894082531875}\Psi_1 \\
C_{13} &= -\frac{130379}{3380195}\Psi_0 + \frac{1434754185803}{498068775579375}\Psi_1 \\
C_{14} &= \frac{499}{7429}\Psi_0 - \frac{2097800621081153}{30951416768146875}\Psi_1 \\
C_{15} &= \frac{27779}{170085}\Psi_0 - \frac{1238550874460233}{101438487632356875}\Psi_1.
\end{aligned} \tag{5.2}$$

Cumulants up to the tenth order agree with those given in Ref. [30]. To our knowledge, the exact numerical values for the eleventh and higher orders have not been so far reported.

6. Discussion

In this Master's Thesis we studied electron and heat transport in low dimensional quantum structures. We calculated the first three conductance cumulants through the superconducting quantum wire. We also calculated first 15 current cumulants through the same structure. These calculations involved the evaluation of an integral with n -point correlation function in the integrand. We proposed a new approach to evaluate these integrals directly, which seemed to work at least in the case $n=1$. In the case $n > 1$ calculations contained multidimensional integrals, which however, at least in the case $n = 2$ seemed to give vanishing contribution.

The reason for this apparent paradox remains still unclear. It is not obvious however, how well a method presented in section 4.1 to calculate the kernel $K(x, y; t)$ suits actually to handle ballistic degenerate initial conditions, because in that case the Lagrange interpolation polynomial, which emerges in those calculations, diverges. So it is possible, that in order to calculate n -point correlation functions for $n > 1$, some extension to the theory is needed.

The nonanalytical length dependence of the third conductance cumulant on the wire length produces an argument for the hypothesis, that the conductance distribution is essentially Gaussian, i.e. all analytical components of the cumulants greater than two vanish, which we now state as a conjecture. In Ref. [30] it was proved, that this is indeed the case for the conductance cumulants up the sixth order. It remains to show, that this statement remains true for cumulants of the

every order.

References

- 1 P. A. Mello and A. D. Stone. *Phys. Rev. B*, 44:3559, 1991.
- 2 Y. Imry and R. Landauer. *Rev. Mod. Phys.*, 71:S306, 1999.
- 3 D. K. Ferry and S. M. Goodnick. *Transport in Nanostructures*. Cambridge University Press, 1997.
- 4 L. I. Schiff. *Quantum Mechanics*. McGraw-Hill, 1968.
- 5 S. Datta. *Electronic Transport in Mesoscopic Systems*. Cambridge University Press, 1995.
- 6 P. W. Brouwer et. al. *Phys. Rev. Lett.*, 85:1064, 2000.
- 7 L. P. Gor'kov et. al. *JETP Lett.*, 30:228, 1979.
- 8 T. Imamura and K. Hikami. *J. Phys. Soc. Jpn.*, 70:3312, 2001.
- 9 R. J. Elliot. *Phys. Rev.*, 96:266, 1954.
- 10 J. D. Bjorken and S. D. Drell. *Relativistic Quantum Mechanics*. McGraw-Hill, 1964.
- 11 M. L. Mehta. *Random Matrices and the Statistical Theory of Energy Levels*. Academic Press, 1967.
- 12 C. W. J. Beenakker. *Rev. Mod. Phys.*, 69:731, 1997.
- 13 E. P. Wigner. *Ann. Math.*, 67:325, 1958.
- 14 F. J. Dyson. *J. Math. Phys.*, 3:140, 1962.
- 15 A. Altland and M. R. Zirnbauer. *Phys. Rev. B*, 55:1142, 1997.
- 16 J. J. M. Verbaarschot. *Nucl. Phys. Proc. Suppl.*, 53:88, 1997.

- 17 S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces*. Academic Press, 1978.
- 18 O. N. Dorokhov. *JETP Lett.*, 36:318, 1982.
- 19 P. A. Mello et. al. *Ann. Phys. (N. Y.)*, 181:290, 1988.
- 20 A. M. S. Macêdo and J. T. Chalker. *Phys. Rev. B*, 46:14985, 1992.
- 21 P. W. Brouwer et. al. *Phys. Rev. Lett.*, 81:862, 1998.
- 22 J. Jacod and P. Potter. *Probability essentials*. Springer-Verlag, 2003.
- 23 M. P. V. Stenberg. *Quantum Coherence and Mesoscopic Fluctuations*. Ph.D. Thesis, Helsinki University of Technology, 2007.
- 24 H. Lee et al. *Phys. Rev. B*, 51:4079, 1995.
- 25 A. M. S. Macêdo. *Phys. Rev. B*, 65:132510, 2002.
- 26 A. F. Macedo-Junior and A. M. S. Macêdo. *J. Phys. A: Math. Theor.*, 41:015004, 2008.
- 27 I. S. Gradshteyn and I. M. Ryzhik. *Tables of Integrals, Series, and Products*. Academic Press, 2007.
- 28 M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. National Bureau of Standards, 1972.
- 29 T. Koelink. *Proc. Am. Math. Soc*, 124:887, 1996.
- 30 M. P. V. Stenberg and J. Särkkä. *Phys. Rev. B*, 74:035327, 2006.